Commutative Algebra
Fall 2013, Lecture 8

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1 Finitely Generated Modules over PIDs

Recall the following definition and the lemma from last lecture:
Definition. A submodule N of M is *pure* if whenever $ax \in N$, with $x \in M, a \in R$, then there exists $z \in N$ such that $az = ax$.

Lemma 1 *If $P = Rx_0$ is a pure cyclic submodule of a finitely generated module N and N/P is a direct sum of cyclic modules then $N = N/P \oplus P$.*

Throughout the lecture today we assume that R is a PID, and M is a finitely generated module over R.
Definition. For p a prime in R we define $M_p = \{m \in torM : \exists i, p^i m = 0\}$.

Proposition 1 *M_p is a direct sum of cyclic modules.*

Proof. Let $x_1, \ldots, x_k$ be a minimum set of generators of M_p. We prove the proposition by induction on k. If $k = 1$ then trivially M is cyclic.

Now suppose $k > 1$. M_p/Rx_1 is generated by $x_2, \ldots, x_k$. So, by induction it is a direct sum of cyclic modules. If Rx_1 is pure then we are done by the lemma, so we just need to show that Rx_1 is pure. Let $R_{p^{n_i}} = Ann_R(x_i)$, in particular $p^{n_i}x_i = 0$. Let $n = \max\{n_1, \ldots, n_k\}$. Permuting if necassary, we may assume without loss of generality that $n_1 = n$. Also $p^n M_p = 0$ since p^n annihilates each generator. Take $y \in M_p, a \in R$ with $ay \in Rx_1$. If $ay = 0$ then $ay = a0$, so the purity condition is satisfied. Suppose $ay \neq 0$. Write $ay = bx_1$ for some $b \in R, b \neq 0$.

Write $a = p^k s, b = p^m t, p \nmid s, p \nmid t$. Since $\gcd(s, p^n) = 1, \exists r, c \in R$ such that $rs + cp^n = 1$, so $rs = 1$ on M_p. As $bx_1 \neq 0$, and $bx_1 = p^m t x_1$ we have $m < n$.

$$\begin{aligned} p^{\overbrace{n-m-1}^{\geq 0}} ay &= p^{n-m-1+k} sy \\ &= p^{n-1} t x_1 \neq 0, \text{ as } p^{n-1} \notin Ann_R x_1. \end{aligned}$$

So $n-m-1+k<n$, so $k<m+1$ so $k \leq m$. So

$$\begin{aligned} ay = p^k sy = p^m tx_1 &= p^m srtx_1, \text{ as } sr = 1 \\ &= ap^{m-k} rtx_1, \end{aligned}$$

where $p^{m-k}rtx_1 \in Rx_1$, as $p^{m-k}rt \in R$. So Rx_1 is pure. □

2 Composition Series

Definition. A module is *simple* if it has no proper nontrivial submodules.
Note. Simple modules are cyclic, as $M = 0$, or take $a \neq 0, a \in M$; Ra is a submodule, so $Ra = M$.
Note. A simple vector space is a 1-dimensional vector space.

Proposition 2 *A nozero module M is simple iff $M \cong R/L$ when L is a maximal left ideal of R.*

Proof. First assume M is simple, we know that M is cyclic, so $M \cong R/L$, where L is a left ideal. If L is not maximal then take $a \in R$, where the ideal generated by L and a, $L+a$, is not R, then $L+a$ gives a proper submodule of M.

Now assume that $M \cong R/L$, where L is a maximal left ideal. Say N is a proper nontrivial submodule of R/L. Take $0 \neq a+L \in N$, then the ideal generated by L and a properly contains L, so by maximality it must be R. So there exist $l \in L, r \in R$ such that $ra + l = 1$, so $r(a+L) = 1 + L \in N$, so N is not proper. □
Definition. Consider a finite descending chain of submodules of M, $M = M_0 \supsetneq M_1 \supsetneq \ldots \supsetneq M_t$. We say the chain has *length* t. The *factors* are the M_i/M_{i+1}, and a *composition series* is a chain with $M_t = 0$ and with all factors simple.
Note.

1. If $M = M_0 \supsetneq M_1 \supsetneq \ldots \supsetneq M_t$ is such a chain , and N a submodule of M_t, then $M/N = M_0/N \supsetneq M_1/N \supsetneq \ldots \supsetneq M_t/N$, and by Second Isomorphism Theorem
$$M_{i-1}/M_i \cong {}^{M_{i-1}/N}\big/_{M_i/N},$$
so the factors are isomorphic.

2. If $M_{i-1} \supsetneq M_i$ and M_{i-1}/M_i is not simple, then there exists N submodule of M_{i-1} such that $M_{i-1} \supsetneq N \supsetneq M_i$, this is called *refining* the chain.

3. If S is a simple submodule of M and N any submodule of M, by Third Isomorphism Theorem $(N+S)/N \cong S/N \cap S$, which is 0 or S. So if $M = \sum_{i=1}^k S_i$, let $M_k = \sum_{i=1}^{t-k} S_i$, then $M = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_{t-1} \supseteq 0$, and, discarding duplicates, this is composition series.

Definition. Two chains are *equivalent* if they are the same length and they have isomorphic factors up to permutations.

Theorem 1 (Schriever-Jordan-Hölder) *Suppose M has a composition series, $M = M_0 \supsetneq M_1 \supsetneq \ldots \supsetneq M_t = 0$, then*

1. *Any finite chain of submodules $M = N_0 \supsetneq N_1 \supsetneq \ldots \supsetneq N_{k-1} \supsetneq 0$ can be refined to be equivalent to $M = M_0 \supsetneq M_1 \supsetneq \ldots \supsetneq M_t$. Consequently*

2. *Any two composite series of M are equivalent.*

3. *Let $l(M)$ be the length of any composition series of M, then $l(M) = l(N) + l(M/N)$ for any N submodule of M, and, in particular, N and M/N have composition series.*

Proof.

1. Let $N_k = 0$, and let $N_{i,j} = N_{i+1} + (M_j \cap N_i), 0 \leq j \leq t, 0 \leq i \leq k-1$. Each $N_{i,j}$ is a submodule of M. Note that

$$\begin{aligned} N_{i,0} &= N_{i+1} + (M \cap N_i) = N_i, \text{ since } N_{i+1} \subseteq N_i \\ N_{i,t} &= N_{i+1} + (0 \cap N_i) = N_{i+1} \end{aligned}$$

So we have

$$\begin{aligned} M = N_0 \supseteq \ldots \supseteq & N_{i-1,t} = N_i = N_{i,0} \supseteq N_{i,1} \\ & \supseteq \ldots \supseteq N_{i,t} = N_{i+1} = N_{i+1,0} \supseteq \ldots \supseteq N_k = 0 \end{aligned}$$

Note many quotients will be 0. We can do the same with the roles of M_i and N_i swapped. Between

$$M_i = M_{i,0} \supseteq M_{i,1} \supseteq \ldots \supseteq M_{i,k} = M_{i+1}$$

all quotients are trivial except one which is M_i/M_{i+1}, since M_i/M_{i+1} is simple.

Claim. $N_{i,j}/N_{i,j+1} \cong M_{j,i}/M_{j,i+1}$. □

Proving the claim suffices to prove (1), because then removing terms which are equal to $M_{i,j}$ sequence is the original composition sequence and the $N_{i,j}$ is equivalent so it is a refinement satisfying (1).

Proof of Claim. We will prove

$$N_{i,j}/N_{i,j+1} \cong N_i \cap M_j / (N_i \cap M_{j+1}) + (N_{i+1} \cap M_j) \cong M_{j,i}/M_{j,i+1}.$$

As the middle term above is symmetric in i, j, it suffices to prove the first equivalence. Third Isomorphism Theorem says $A+B/B \cong A/A\cap B$. Let $A = N_i \cap M_j, B = N_{i,j+1} = N_{i+1} + (M_{j+1} \cap N_i)$, then

$$\begin{aligned} A + B &= (N_i \cap M_j) + N_{i+1} + (M_{j+1} \cap N_i) \\ &= N_{i+1} + (N_i \cap M_j) \text{ since } M_{j+1} \subseteq M_j \\ &= N_{i,j}. \end{aligned}$$

And

$$\begin{aligned} A \cap B &= (N_i \cap M_j) \cap (N_{i,j+1} = N_{i+1} + (M_{j+1} \cap N_i)) \\ &= (N_{i+1} \cap M_j) + (M_{j+1} \cap N_i). \end{aligned}$$

■

2. Direct consequence of (1).
3. Refine $M \supseteq N \supseteq 0$ to a composition series. This givesa composition series for N (just truncate) and it gives $M = M_0 \supsetneq M_1 \supsetneq \ldots \supsetneq M_v = N$, then

$$M/N = M_0/N \supsetneq M_1/N \supsetneq \ldots \supsetneq M_v/N = N/N = 0,$$

and so the lengths add.

□

Corollary 1 *Say N is a submodule of M, $l(N) = l(M) < \infty$. Then $M = N$.*

Proof. $l(M/N) + l(N) = l(M)$, so $l(M/N) = 0$, so $M/N = 0$, so $M = N$. □

Corollary 2 *If $0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$ is an exact sequence of modules, then $l(M) = l(K) + l(N)$.*

Proof. By exactness we have $l(K) = l(f(K))$. Also, by First Isomorphism Theorem, and then, by exactness, we have $N \cong M/\ker g = M/f(K)$, so

$$l(K) + l(N) = l(f(K)) + l(M/f(K)) = l(M).$$

□

Note. The proof of Jordan-Hölder Theorem only needs the three isomorphism theorems, so it holds in any context where they all are true, including groups.